GCSE Maths — Vectors

Notes by Dennis Mihailov

1. What is a Vector?

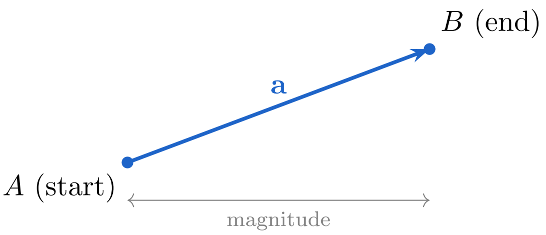
Definition. A **vector** is a quantity that has both a **magnitude** (size) and a **direction**.

A **scalar** is a quantity that has only magnitude (no direction).

Scalars		Vectors	
Speed	60 mph	**Velocity**	60 mph north
Distance	5 km	**Displacement**	5 km east
Mass	70 kg	**Force**	10 N upward
Temperature	20°C	**Acceleration**	9.8 m/s² downward

Drawing a vector

A vector is drawn as an arrow. The **length** of the arrow represents the magnitude and the **arrowhead** shows the direction.



Notation

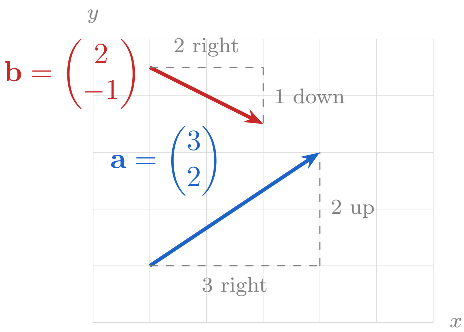
There are three common ways to write vectors at GCSE:

- **Bold letter:** **a**, **b**, **c** (used in textbooks)
- **Underlined letter:** $\underline{a}$, $\underline{b}$ (used in handwriting)
- **Column vector:** $\begin{pmatrix} x \\ y \end{pmatrix}$ where x = horizontal, y = vertical
- **Between two points:** $\overrightarrow{AB}$ (the vector from A to B)

Column vectors on a grid

A column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ means: move x units **right** and y units **up**.

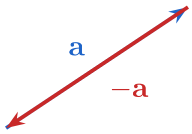
Negative values mean left / down.



The negative of a vector

$-\mathbf{a}$ is the same vector as $\mathbf{a}$ but pointing in the **opposite direction**.

$$\text{If } \mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \text{then } -\mathbf{a} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$



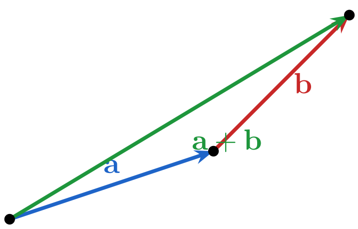
2. Adding and Subtracting Vectors

Adding vectors

To add two column vectors, add the top numbers together and the bottom numbers together.

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

Geometrically: place the second vector at the end of the first. The result is the arrow from the start of the first to the end of the second (the shortcut).



Example 1. Let $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Find $\mathbf{a} + \mathbf{b}$.

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 + (-1) \\ 1 + 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Subtracting vectors

Subtracting is the same as adding the negative:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$$

Example 2. Let $\mathbf{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Find $\mathbf{a} - \mathbf{b}$.

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 5 - 3 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

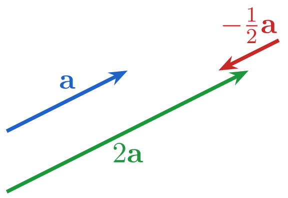
Watch out! Order matters in subtraction. $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$.
Always subtract the second from the first, component by component.

3. Multiplying a Vector by a Scalar

Scalar multiplication stretches (or shrinks) a vector. It changes the **magnitude** but **not the direction** (unless the scalar is negative, in which case the direction reverses).

$$k\mathbf{a} = k \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k\, a_1 \\ k\, a_2 \end{pmatrix}$$

Multiply each component by the scalar k.



half the size, reversed

Example 3. Let $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Find $3\mathbf{a}$ and $-2\mathbf{a}$.

$$3\mathbf{a} = 3 \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \end{pmatrix} \qquad -2\mathbf{a} = -2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

Parallel vectors

Two vectors are **parallel** if one is a scalar multiple of the other.

a and **b** are parallel $\iff$ $\mathbf{b} = k\mathbf{a}$ for some scalar k.

Example 4. Are $\mathbf{a} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ parallel?

Check: is there a k such that $\mathbf{b} = k\mathbf{a}$?

$$k = \frac{6}{4} = 1.5 \quad \text{and} \quad k = \frac{9}{6} = 1.5 \quad \checkmark$$

Yes — $\mathbf{b} = 1.5\,\mathbf{a}$, so the vectors are parallel.

4. Vector Paths and Geometry Proofs

Finding a vector path

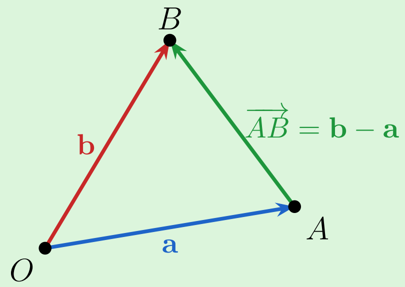
If you need to get from one point to another, you can travel along **known** vectors, possibly in reverse.

Key rule: $\overrightarrow{AB} = -\overrightarrow{BA}$

To go from A to C via B: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Example 5. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Find $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$



Midpoints

The **midpoint** M of AB lies halfway along $\overrightarrow{AB}$, so:

$$\overrightarrow{OM} = \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB}$$

Example 6. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB. Find $\overrightarrow{OM}$.

$$\overrightarrow{OM} = \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}\mathbf{a} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}$$

Geometry proofs

Vectors are powerful for **proving** geometric facts. The key ideas are:

- If $\overrightarrow{PQ} = k\overrightarrow{RS}$, then PQ is **parallel** to RS.
- If two vectors share a point and are parallel, the three points are **collinear** (on a straight line).
- The **ratio** of $|k|$ tells you how many times longer one segment is than another.

Example 7 (Proof). $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.
P divides OA in ratio $1 : 1$ (midpoint of OA).
Q divides OB in ratio $1 : 1$ (midpoint of OB).
Prove that PQ is parallel to AB and half its length.

Step 1: Find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

$$\overrightarrow{OP} = \tfrac{1}{2}\mathbf{a}, \qquad \overrightarrow{OQ} = \tfrac{1}{2}\mathbf{b}$$

Step 2: Find $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b} = \tfrac{1}{2}(\mathbf{b} - \mathbf{a})$$

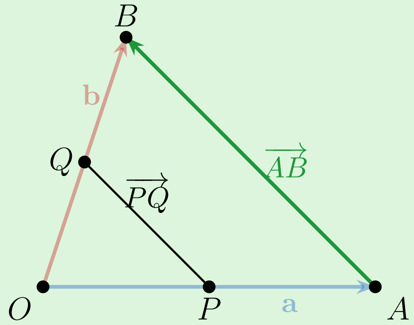
Step 3: Find $\overrightarrow{AB}$.

$$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

Conclusion:

$$\overrightarrow{PQ} = \tfrac{1}{2}\,\overrightarrow{AB}$$

Since $\overrightarrow{PQ}$ is a scalar multiple of $\overrightarrow{AB}$, the lines are **parallel**, and PQ is **half the length** of AB. □



Points on a line (ratio problems)

A common exam question gives a point that divides a line in a given ratio.

If P divides AB in the ratio $m : n$, then:

$$\overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$$

Example 8. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.
P divides AB in ratio $2 : 1$. Find $\overrightarrow{OP}$.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$
$$\overrightarrow{AP} = \tfrac{2}{3}\overrightarrow{AB} = \tfrac{2}{3}(\mathbf{b} - \mathbf{a})$$
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \tfrac{2}{3}(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \tfrac{2}{3}\mathbf{b} - \tfrac{2}{3}\mathbf{a} = \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\mathbf{b}$$

5. Summary of Key Rules

Adding column vectors	Add components: $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$
Subtracting column vectors	Subtract components: $\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$
Scalar multiplication	Multiply each component: $k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$
Negative vector	Reverse direction: $-\mathbf{a} = (-1)\mathbf{a}$
Parallel vectors	$\mathbf{b} = k\mathbf{a}$ for some scalar k
Path between points	$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
Reversing a vector	$\overrightarrow{BA} = -\overrightarrow{AB}$
Midpoint M of AB	$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$
Ratio $m : n$ division	$\overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$

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